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The Jacobi map

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Abstract

This paper defines nth order Jacobi fields to be solutions to a second-order nonlinear differential equation defined by the Jacobi map. nth order Jacobi fields arise naturally as acceleration vector fields of geodesic variations. As a main theorem we prove necessity and sufficiency conditions for an nth order Jacobi field to be the acceleration vector field of a variation of geodesics normal to a submanifold. An m geodesic, $m \geq 2$, is a smooth curve whose mth covariant derivative vanishes. We prove an index theorem giving bounds for the total m focal multiplicity along an m geodesic m normal to a submanifold in a flat manifold.

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1. Introduction

Using two-parameter variations in a smooth semi-Riemannian manifold (M, g) we prove the existence and uniqueness of the Jacobi map

$$F_n : \bigoplus_{i=1}^{3n-1} TM \to TM$$

from the $3n - 1$ fold Whitney sum of TM to TM, see Lemma 2.1. It gives rise to a second-order possibly nonlinear differential equation in a vector field along a geodesic, see Definition 2.3. Solutions to this differential equation are called nth order Jacobi vector fields, since they arise naturally as the acceleration vector fields of geodesic variations. If there are consecutive zeroes of an nth order Jacobi field Y^n along γ, $Y^n(0) = 0$, $Y^n(a) = 0, a > 0$ it means that there are geodesics emanating from $\gamma(0)$ which meet to nth order at $\gamma(a)$.

This makes the results applicable in physics. For instance, it applies to the motion of a particle on a surface in a potential, because here the equations of motion are the geodesics

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of the Jacobi metric. We are thus able to detect the occurrence of almost meeting points for orbits of the equations of motion.

The main theorem in Section 3 proves necessity and sufficiency conditions for an nth order Jacobi field to be the variation vector field of a variation through normal geodesics to a submanifold P. If an nth order Jacobi field Y^n along the geodesic γ satisfies these conditions at $\gamma(0) = p$ and has at $a > 0$ a zero $Y^n(a) = 0$ then using Theorem 3.2 it follows that there are geodesics normal to P meeting at $\gamma(a)$ up to nth order. This is the geometrical significance of Theorem 3.2. Section 4 shows that nth order Jacobi fields describe the nth derivative of the exponential map.

At each $p \in M$ there is a unique m geodesic $\alpha : I \to M$ having ith acceleration $v_i \in T_pM, i = 1, \dots, m-1$, see Proposition 5.2. m geodesic variations give rise to m Jacobi fields as variation vector fields. These m Jacobi fields are solutions to an mth order linear differential equation. So m Jacobi fields Y are determined by m initial velocities

$$v_i = Y^{(i)}(0), \quad i = 0, \dots, m-1.$$

Unlike the situation with Jacobi fields there are always nonzero m Jacobi fields, $m \geq 3$, which has consecutive zeroes, see Proposition 5.6. The second main theorem is the index Theorem 5.10. It gives upper and lower bounds for the focal multiplicity along an m geodesic $\gamma : I \to M$, $\mathbb{R}_+ \subset I$, m normal to a submanifold P of a flat Riemannian manifold M. More precisely it is proven that the space of tangential m Jacobi fields arising as variation vector fields of variations of γ through m geodesics m normal to P and vanishing at $b > 0$ is a vector space $V(b)$. The dimension of $V(b)$ is less than or equal to the algebraic multiplicity of b and

$$\text{index} - S_{\gamma^{(m-1)}(0)} \leq \sum_{b \in \mathbb{R}^+} \dim V(b),$$

where $S_{\gamma^{(m-1)}(0)}$ is the shape operator of P relative to $\gamma^{(m-1)}(0)$.

2. nth order Jacobi fields

Let $\sigma : I \times J \to M$ denote a smooth two-parameter variation. Define

$$\begin{aligned} r_\sigma(s,t) = \Bigg\{ & \sum_{j=0}^{n-1} (R(\sigma_s, \sigma_t)\sigma_{s^j t})_{s^{n-1-j}} \\ & + \sum_{j=1}^{n-1} (R(\sigma_s, \sigma_t)\sigma_{s^j})_{ts^{n-1-j}} - R(\sigma_{s^n}, \sigma_t)\sigma_t \Bigg\} (s,t). \end{aligned}$$

We start with the definition of the Jacobi map F_n.

Lemma 2.1. *There exists a unique, fibre preserving, smooth map*

$$F_n : \bigoplus_{i=1}^{3n-1} TM \to TM$$

such that

$$r_\sigma(0,0) = F_n((\sigma_s, \ldots, \sigma_{s^{n-1}}, \sigma_t, \ldots, \sigma_{s^{n-1}t}, \sigma_{tt}, \ldots, \sigma_{tts^{n-1}})(0,0))$$

for every smooth two-parameter variation σ.

Proof. We claim that

$$\sigma_{s^i t s^j} = \sigma_{s^{i+j}t} + \sum_{k=0}^{j-1} (R(\sigma_t, \sigma_s)\sigma_{s^{i+k}})_{s^{j-1-k}}$$

for all $i, j \geq 1$. For $j = 1$ we have

$$\sigma_{s^i ts} = \sigma_{s^{i+1}t} + R(\sigma_t, \sigma_s)\sigma_{s^i}.$$

So the claim is true for $i \in \mathbb{N}$ and $j = 1$. Now fix $i \in \mathbb{N}$ and assume the validity of the claim for $j \in \mathbb{N}$. Then

$$\begin{aligned}
\sigma_{s^i t s^{j+1}} = (\sigma_{s^i t s^j})_s &= \left(\sigma_{s^{i+j}t} + \sum_{k=0}^{j-1} (R(\sigma_t, \sigma_s)\sigma_{s^{i+k}})_{s^{j-1-k}}\right)_s \\
&= \sigma_{s^{i+j+1}t} + R(\sigma_t, \sigma_s)\sigma_{s^{i+j}} + \sum_{k=0}^{j-1} (R(\sigma_t, \sigma_s)\sigma_{s^{i+k}})_{s^{j-k}} \\
&= \sigma_{s^{i+j+1}t} + \sum_{k=0}^{j+1-1} (R(\sigma_t, \sigma_s)\sigma_{s^{i+k}})_{s^{j-k}}.
\end{aligned}$$

The claim follows.

Our second claim pertains to the existence of a smooth fibre preserving map

$$H_{ij} : \bigoplus_{k=1}^{2(i+j)-1} TM \to TM, \quad i \geq 0, \ j \geq 1,$$

such that $\sigma_{s^i t s^j} = \sigma_{s^{i+j}t} + H_{ij}(\sigma_s, \ldots, \sigma_{s^{i+j-1}}, \sigma_t, \ldots, \sigma_{s^{i+j-1}t})$ for all smooth two-parameter variations σ.

Clearly $H_{01} = 0$. Also $H_{02}(v_1, w_1, w_2) = R(w_1, v_1)v_1$. Assume H_{0k} has been defined for $k = 1, \ldots, j$.

Fix $p \in M$ and let (U, ϕ) denote a chart around p. We can assume $\operatorname{Im}\sigma \subset U$ and let $\tau = \phi \circ \sigma$. Verify by induction that there exist smooth maps

$$h_i : \phi(U) \times (\mathbb{R}^n)^i \to T_3^1 M, \quad k_i : \phi(U) \times (\mathbb{R}^n)^{i-1} \to \mathbb{R}^n, \qquad i \geq 1,$$

such that

$$R_{s^i} \circ \sigma(s,t) = h_i\left(\tau(s,t), \ldots, \frac{\partial^i \tau}{\partial s^i}(s,t)\right),$$

$$\sigma_{s^i}(s,t)^\phi = \frac{\partial^i \tau}{\partial s^i}(s,t) + k_i\left(\tau(s,t), \ldots, \frac{\partial^{i-1}\tau}{\partial s^{i-1}}(s,t)\right).$$

Using the k_i it is easy to see that given $v_1, \ldots, v_i \in T_qM$ for some $q \in M$ there exists a two-parameter variation σ such that $\sigma_s(0,0) = v_1, \ldots, \sigma_{s^i}(0,0) = v_i$. Now define

$$H_i(v_1, \ldots, v_i) = \begin{cases} R \circ \sigma(0,0), & i = 0, \\ R_{s^i} \circ \sigma(0,0), & i > 0. \end{cases}$$

In local coordinates when $q \in U$

$$H_i(v_1, \ldots, v_i) = h_i\left(\phi(q), \ldots, \frac{\partial^i \tau}{\partial s^i}(s,t)\right).$$

Using the k_i we can inductively show the existence of smooth maps

$$l_k : \phi(U) \times (\mathbb{R}^n)^k \to \mathbb{R}^n, \quad 1 \le k \le i,$$

such that

$$\frac{\partial^k \tau}{\partial s^k}(0,0) = l_k(\phi(q), v_1^\phi, \ldots, v_k^\phi),$$

hence

$$H_i(v_1, \ldots, v_i) = h_i(\phi(q), \ldots, l_i(\phi(q), v_1^\phi, \ldots, v_i^\phi)).$$

This shows that H_i is well defined and smooth. Now

$$\begin{aligned}
\sigma_{ts^{j+1}} &= \sigma_{s^{j+1}t} + \sum_{k=1}^{j} (R(\sigma_t, \sigma_s)\sigma_{s^k})_{s^{j-k}} \\
&= \sigma_{s^{j+1}t} + \sum_{k=1}^{j} \sum_{\sum_{i=1}^4 k_i = j-k} R_{s^{k_1}}(\sigma_{ts^{k_2}}, \sigma_{s^{k_3+1}})\sigma_{s^{k+k_4}} \\
&= \sigma_{s^{j+1}t} + \sum_{k=1}^{j} \sum_{\sum_{i=1}^4 k_i = j-k} H_{k_1}(\sigma_s, \ldots, \sigma_{s^{k_1}}) \\
&\qquad (\sigma_{s^{k_2}t} + H_{0k_2}(\sigma_s, \ldots, \sigma_{s^{k_2-1}}, \sigma_t, \ldots, \sigma_{s^{k_2-1}t}), \sigma_{s^{k_3+1}})\sigma_{s^{k+k_4}}.
\end{aligned}$$

So we can define

$$\begin{aligned}
&H_{0(j+1)}(v_1, \ldots, v_j, w_0, \ldots, w_j) \\
&= \sum_{k=1}^{j} \sum_{\sum_{i=1}^4 k_i = j-k} H_{k_1}(v_1, \ldots, v_{k_1}) \\
&\qquad (w_{k_2} + H_{0k_2}(v_1, \ldots, v_{k_2-1}, w_0, \ldots, w_{k_2-1}), w_{k_3+1})w_{k+k_4}.
\end{aligned}$$

So we have defined H_{0j} for all $j \ge 1$.

Using the first claim we see that

$$\sigma_{s^i ts^{j+1}} = \sigma_{s^{i+j+1}t} + \sum_{k=0}^{j} \sum_{\sum_{i=1}^{4} k_i = j-k} H_{k_1}(\sigma_s, \ldots, \sigma_{s^{k_1}})$$
$$(\sigma_{s^{k_2}t} + H_{0k_2}(\sigma_s, \ldots, \sigma_{s^{k_2-1}}, \sigma_t, \ldots, \sigma_{s^{k_2-1}t}), \sigma_{s^{k_3+1}})\sigma_{s^{i+k+k_4}}.$$

Thus

$$H_{ij}(v_1, \ldots, v_{i+j-1}, w_0, \ldots, w_{i+j-1})$$
$$= \sum_{k=0}^{j-1} \sum_{\sum_{i=1}^{4} k_i = j-k-1} H_{k_1}(v_1, \ldots, v_{k_1})$$
$$(w_{k_2} + H_{0k_2}(v_1, \ldots, v_{k_2-1}, w_0, \ldots, w_{k_2-1}), v_{k_3+1})v_{i+k+k_4}.$$

Let $v_1, \ldots, v_k, w_0, \ldots, w_k, x_0, \ldots, x_k \in T_pM$. Arguing as above we can verify the existence of a two-parameter variation σ with

$$\sigma_s(0,0) = v_1, \ldots, \sigma_{s^k}(0,0) = v_k,$$
$$\sigma_t(0,0) = w_0, \ldots, \sigma_{s^k t}(0,0) = w_k,$$
$$\sigma_{tt}(0,0) = x_0, \ldots, \sigma_{tts^k}(0,0) = x_k.$$

As above we can argue that the map

$$H_k^1 : \bigoplus_{i=1}^{2k+1} TM \to T_3^1 M$$

defined by

$$H_k^1(v_1, \ldots, v_k, w_0, \ldots, w_k) = R_{ts^k} \circ \sigma(0,0)$$

is well defined and smooth. Now we can compute

$$\sigma_r(0,0) = \sum_{j=0}^{n-1} \sum_{\sum k_i = n-1-j, k_2 \neq n-1} H_{k_1}(\sigma_s, \ldots, \sigma_{s_{k_1}})$$
$$(\sigma_{s^{k_2+1}}, \sigma_{s^{k_3}t} + H_{0k_3}(\sigma_s, \ldots, \sigma_{s^{k_3-1}t}))\sigma_{s^{j+k_4}t} + H_{jk_4}(\sigma_s, \ldots, \sigma_{s^{j+k_4-1}t})$$
$$+ \sum_{j=1}^{n-1} \sum_{\sum k_i = n-1-j} H_{k_1}^1(\sigma_s, \ldots, \sigma_{s^{k_1}t})$$
$$(\sigma_{s^{k_2+1}}, \sigma_{s^{k_3}t} + H_{0k_3}(\sigma_s, \ldots, \sigma_{s^{k_3-1}t}))\sigma_{s^{j+k_4}}$$
$$+ \sum_{\sum k_i = n-1-j} H_{k_1}(\sigma_s, \ldots, \sigma_{s^{k_1}})$$
$$(\sigma_{s^{1+k_2}t} + H_{1k_2}(\sigma_s, \ldots, \sigma_{s^{k_2}t}), \sigma_{s^{k_3}t} + H_{0k_3}(\sigma_s, \ldots, \sigma_{s^{k_3-1}t}))\sigma_{s^{j+k_4}}$$
$$+ \sum_{\sum k_i = n-1-j} H_{k_1}(\sigma_s, \ldots, \sigma_{s^{k_1}})(\sigma_{s^{k_2+1}}, \sigma_{tts^{k_3}})\sigma_{s^{j+k_4}}$$

$$+ \sum_{\sum k_i = n-1-j} H_{k_1}(\sigma_s, \ldots, \sigma_{s^{k_1}})$$
$$(\sigma_{s^{k_2+1}}, \sigma_{s^{k_3}t} + H_{0k_3}(\sigma_s, \ldots, \sigma_{s^{k_3-1}t}))\sigma_{s^{j+k_4}t} + H_{jk_4}(\sigma_s, \ldots, \sigma_{s^{j+k_4-1}t})$$
$$= F_n((\sigma_s, \ldots, \sigma_{s^{n-1}}, \sigma_t, \ldots, \sigma_{s^{n-1}t}, \sigma_{tt}, \ldots, \sigma_{tts^{n-1}})(0,0)),$$

which shows that F_n is well defined and smooth. □

Corollary 2.2. *For every $j \geq 0$ there exist a unique fibre preserving map*

$$G_j : \bigoplus_{k=1}^{2j+1} TM \to TM$$

such that

$$\sigma_{s^{j+1}t} = \sigma_{ts^{j+1}} + G_j(\sigma_s, \ldots, \sigma_{s^j}, \sigma_t, \ldots, \sigma_{ts^j})$$

for every two-parameter variation $\sigma : I \times J \to M$.

Proof. Just let $G_j = H_{0(j+1)}$ from the proof of Lemma 2.1. □

Now let $\gamma : I \to M$ denote a geodesic in (M, g).

Definition 2.3. An nth order Jacobi field along γ is a map

$$Y^n = (Y_1, \ldots, Y_n) : I \to (TM)^n$$

such that each Y_k is a smooth vector field along γ with

$$Y_k''(t) = R(Y_k, \gamma')\gamma'(t) + F_k((Y_1, \ldots, Y_{k-1}, \gamma', Y_1', \ldots, Y_{k-1}', 0, \ldots, 0)(t))$$

for all $t \in I$ and all $k \in \{1, \ldots, n\}$.

Proposition 2.4. *Given $v_1, \ldots, v_n, w_1, \ldots, w_n \in T_{\gamma(0)}M$ then there exists an nth order Jacobi field Y^n along γ such that*

$$Y_i(0) = v_i, \quad Y_i'(0) = w_i, \qquad i = 1, \ldots, n.$$

If $X^n : I \to (TM)^n$ is an nth order Jacobi field along γ satisfying

$$X_i(0) = v_i, \quad X_i'(0) = w_i, \qquad i = 1, \ldots, n,$$

then $X^n = Y^n$.

Proof. Straightforward. □

The following proposition shows that nth order Jacobi fields arise as the acceleration vector fields of geodesic variations. In fact:

Proposition 2.5. *Let $\sigma : I \times J \to M$ denote a smooth geodesic variation of the geodesic $\gamma : J \to M$ with acceleration vector fields*

$$V_i(t) = \sigma_{s^i}(0, t), \quad i = 1, \dots, n.$$

Then $V^n = (V_1, \dots, V_n)$ is an nth order Jacobi field along γ.

Proof. Assume for $k \geq 1$ that

$$\sigma_{s^k tt} = \sum_{j=0}^{k-1} (R(\sigma_s, \sigma_t)\sigma_{s^j t})_{s^{k-j-1}} + \sum_{j=1}^{k-1} (R(\sigma_s, \sigma_t)\sigma_{s^j})_{ts^{k-1-j}} \tag{2.1}$$

agreeing that a sum with a smaller top index than bottom index is zero. For $k = 1$ this is true by [4, p.123]. But then

$$\begin{aligned}\sigma_{s^{k+1}tt} &= \sigma_{s^k tst} + (R(\sigma_s, \sigma_t)\sigma_{s^k})_t = (\sigma_{s^k tt})_s + R(\sigma_s, \sigma_t)\sigma_{s^k t} + (R(\sigma_s, \sigma_t)\sigma_{s^k})_t \\ &= \sum_{j=0}^{k+1-1} (R(\sigma_s, \sigma_t)\sigma_{s^j t})_{s^{k-j}} + \sum_{j=1}^{k+1-1} (R(\sigma_s, \sigma_t)\sigma_{s^j})_{ts^{k-j}}.\end{aligned}$$

Now (2.1) follows by induction. Finally

$$\sigma_{s^k tt}(0, t) = R(\sigma_{s^k}, \sigma_t)\sigma_t + F_k((\sigma_s, \dots, \sigma_{s^{k-1}}, \sigma_t, \dots, \sigma_{s^{k-1}t}, 0, \dots, 0)(0, t)).$$

The proposition follows. □

3. Endmanifolds

Now let P denote a semi-Riemannian submanifold of M^m.

Lemma 3.1. *There exists a smooth map*

$$\Omega_j : \begin{cases} \bigoplus^{j-1} TP \to T_P M, & j \geq 2, \\ TP \to T_P M, & j = 1, \end{cases}$$

such that

$$\operatorname{nor} \alpha^{(j)} = \Omega_j(\alpha', \dots, \tan \alpha^{(j-1)})$$

for all smooth curves $\alpha : J \to P \subset M$. Here $\alpha^{(k)}$ denotes k times induced covariant differentiation in M of α.

Proof. Clearly $\Omega_1 = 0$ and $\Omega_2(\alpha') = \Pi(\alpha', \alpha')$, where Π is the second fundamental form of P.

Take a chart (U, ϕ) on M around some $p \in P$ such that

$$P \cap U = \{q \in U \mid \phi_{k+1}(q) = 0, \dots, \phi_m(q) = 0\},$$

$$\operatorname*{span}_{i=k+1,\dots,m} \{\partial_i(q)\} = T_q P^{\perp}, \qquad q \in U \cap P.$$

Verify by induction that there exist smooth functions

$$h_i : \phi(U) \times (\mathbb{R}^n)^{i-1} \to \mathbb{R}^n$$

such that

$$\alpha^{(i)\phi} = \frac{\mathrm{d}^i(\phi \circ \alpha)}{\mathrm{d}v^i} + h_i\left(\phi \circ \alpha, \dots, \frac{\mathrm{d}^{i-1}(\phi \circ \alpha)}{\mathrm{d}v^{i-1}}\right).$$

Given $v_1, \dots, v_j \in T_q P$ then there exists a smooth curve $\alpha : J \to P$ with

$$\tan \alpha^{(i)}(0) = v_i, \quad i = 1, \dots, j.$$

Define

$$\Omega_j(v_1, \dots, v_j) = \operatorname{nor} \alpha^{(j)}(0). \tag{3.1}$$

Using the h_i verify the existence of smooth maps

$$k_i : \phi(U) \times (\mathbb{R}^n)^{i-1} \to \mathbb{R}^n$$

such that

$$\frac{\mathrm{d}^i(\phi \circ \alpha)}{\mathrm{d}v^i} = k_i(\phi \circ \alpha, \alpha'^{\phi}, \dots, \tan \alpha^{(i)\phi}).$$

In coordinates (3.1) becomes

$$\begin{aligned}\Omega_j(v_1, \dots, v_j)^{\phi} &= \operatorname{nor} h_j\left(\phi \circ \alpha, \dots, \frac{\mathrm{d}^{j-1}(\phi \circ \alpha)}{\mathrm{d}v^{j-1}}\right)\\ &= \operatorname{nor} h_j(\phi(q), v_1^{\phi}, \dots, k_{j-1}(\phi(q), \dots, v_{j-1}^{\phi})),\end{aligned}$$

which shows that the Ω_j are well defined and smooth. The lemma follows. □

Now let $\alpha : I \to P$ denote a smooth curve in P with

$$\alpha^{(i)}(0) = V_i(0), \quad i = 1, \dots, n,$$

where $(V_1, \dots, V_n)$ is an nth order Jacobi field along the geodesic $\gamma : J \to M$, normal to P at 0. Here J is a closed interval. Define inductively normal parallel vector fields $A_1, \dots, A_n$ along α. Let A_1 be the normal parallel vector field along α with $A_1(0) = \gamma'(0)$. When A_i has been defined let A_{i+1} be the normal parallel vector field along γ with

$$\begin{aligned}A_{i+1}(0) = \operatorname{nor}\Bigg\{ & V_i'(0) - G_{i-1}(V_1(0), \dots, V_{i-1}(0),\\ & \gamma'(0), V_1'(0), \dots, V_{i-1}'(0)) - \sum_{l=0}^{i-1} K_{i,l} A_{l+1}^{(i-l)}(0)\Bigg\}.\end{aligned}$$

Theorem 3.2. *An nth order Jacobi field $V^n = (V_1, \ldots, V_n)$ along a geodesic $\gamma : J \to M$ normal to P at 0 is the acceleration vector field of a smooth variation σ through normal geodesics with initial curve α iff*

$$\begin{aligned}&\tan\{V_i'(0) - G_{i-1}(V_1(0), \ldots, V_{i-1}(0), \gamma'(0), \ldots, V_{i-1}'(0))\}\\ &\quad = \tan\left\{\sum_{l=0}^{i-1} K_{i,l} A_{l+1}^{(i-l)}(0)\right\},\\ &\operatorname{nor} V_i(0) = \Omega_i(\tan V_1(0), \ldots, \tan V_{i-1}(0)), \qquad i = 1, \ldots, n.\end{aligned} \tag{3.2}$$

Remark 3.3. An nth order Jacobi field V^n satisfying (3.2) is called an nth order P Jacobi field along γ.

Proof. Let $\sigma : I \times J$ denote a smooth variation of geodesics normal to P with initial curve α. Then

$$\alpha(v) = \sigma(v, 0) \in P, \qquad v \in I.$$

We have seen that

$$\begin{aligned}\operatorname{nor} V_i(0) &= \operatorname{nor} \alpha^{(i)}(0) = \Omega_i(\tan \alpha'(0), \ldots, \tan \alpha^{(i-1)}(0))\\ &= \Omega_i(\tan V_1(0), \ldots, \tan V_{i-1}(0)).\end{aligned}$$

Take a normal parallel basis $E_{k+1}, \ldots, E_m$ along α. We know that

$$\sigma_t(v, 0) = \sum_{j=k+1}^{m} g_j(v) E_j(v)$$

for some smooth functions

$$g_j : I \to \mathbb{R}, \qquad i = k+1, \ldots, m.$$

Leibnitz' rule gives

$$\sigma_{ts^i}(v, 0) = \sum_{j=k+1}^{m} \sum_{s=0}^{i} K_{i,s} g_j^{(s)}(v) E_j^{(i-s)}(v). \tag{3.3}$$

For $i = 1$

$$\tan V_1'(0) = \sum_{j=k+1}^{m} g_j(0) \tan E_j'(0).$$

Since

$$\sigma_t(0, 0) = \sum_{j=k+1}^{m} g_j(0) E_j(0) = \gamma'(0)$$

we find $\tan V_1'(0) = \tan A_1'(0)$. We claim that

$$\sum_{j=k+1}^{m} g_j^{(i)}(0)E_j(0) = \text{nor}\left\{V_i'(0) - G_{i-1}(V_1(0), \dots, V_{i-1}(0), \gamma'(0), \dots, V_{i-1}'(0)) - \sum_{l=0}^{i-1} K_{i,l}\, \text{nor}\, A_{l+1}^{(i-l)}(0)\right\} = A_{i+1}(0)$$

for all $i = 1, \dots, n$. For $i = 1$ this is

$$\sum_{j=k+1}^{m} g_j'(0)E_j(0) = \text{nor}\, V_1'(0) - \text{nor}\, A_1'(0) = \text{nor}\, V_1'(0) = A_2(0).$$

Assuming the claim is true for all $i < i_* \le n$ we find

$$\begin{aligned}\text{nor}\, V_{i_*}'(0) = \text{nor}\{\sigma_{ts^{i_*}}(0,0)\} &= \text{nor}\left\{V_{i_*}'(0) - \sum_{l=1}^{i_*-1}(R(\sigma_s, \sigma_t)\sigma_{s^l})_{s^{i_*-1-l}}\right\} \\ &= \text{nor}\left\{\sum_{j=k+1}^{m}\sum_{i=0}^{i_*} K_{i_*,i}\, g_j^{(i)}(0)E^{(i_*-i)}(0)\right\}\end{aligned}$$

by (3.3). Hence

$$\begin{aligned}&\text{nor}\{V_{i_*}'(0) - G_{i_*-1}(V_1(0), \dots, V_{i_*-1}(0), \gamma'(0), \dots, V_{i_*-1}'(0))\} \\ &= \text{nor}\sum_{j=k+1}^{m} g_j^{(i_*)}(0)E_j(0) + \text{nor}\sum_{j=k+1}^{m}\sum_{i=0}^{i_*-1} K_{i_*,i}\, g_j^{(i)}(0)E_j^{(i_*-i)}(0) \\ &= \sum_{j=k+1}^{m} g_j^{(i_*)}(0)E_j(0) + \sum_{i=0}^{i_*-1} K_{i_*,i}\, \text{nor}\, A_{i+1}^{(i_*-i)}(0)\end{aligned}$$

and from this the claim follows. Finally

$$\begin{aligned}\tan \sigma_{ts^i}(0,0) &= \tan\{V_i'(0) - G_{i-1}(V_1(0), \dots, V_{i-1}(0), \gamma'(0), \dots, V_{i-1}'(0))\} \\ &= \sum_{j=k+1}^{m}\sum_{s=0}^{i} K_{i,s}\, g_j^{(s)}(0) \tan E_j^{(i-s)}(0) = \sum_{s=0}^{i-1} K_{i,s} \tan A_{s+1}^{(i-s)}(0).\end{aligned}$$

To prove the converse statement let $V^n = (V_1, \dots, V_n)$ denote an nth order Jacobi field along γ satisfying (3.2).

We claim that there exists a normal vector field Z along α such that

$$\begin{aligned}&Z(0) = \gamma'(0), \\ &Z^{(i)}(0) = V_i'(0) - G_{i-1}(V_1(0), \dots, V_{i-1}(0), \gamma'(0), \dots, V_{i-1}'(0)), \qquad i = 1, \dots, n.\end{aligned}$$

Define

$$Z(v) = \sum_{i=1}^{n} \frac{1}{(i-1)!} v^{i-1} A_i(v).$$

Then $Z(0) = \gamma'(0)$ and

$$
\begin{aligned}
Z^{(i)}(0) &= \sum_{l=0}^{i} K_{i,l} A_{l+1}^{(i-l)}(0) \\
&= A_{i+1}(0) + \operatorname{nor} \sum_{l=0}^{i-1} K_{i,l} A_{l+1}^{(i-l)}(0) + \tan \sum_{l=0}^{i-1} K_{i,l} A_{l+1}^{(i-l)}(0) \\
&= \operatorname{nor}\{V_i'(0) - G_{i-1}(V_1(0), \ldots, V_{i-1}(0), \gamma'(0), \ldots, V_{i-1}'(0))\} \\
&\quad + \tan\{V_i'(0) - G_{i-1}(V_1(0), \ldots, V_{i-1}(0), \gamma'(0), \ldots, V_{i-1}'(0))\}.
\end{aligned}
$$

The claim follows.

Now define

$$\sigma(s,t) = \exp(tZ(s)), \qquad (s,t) \in J \times I$$

by shrinking J if necessary. Define acceleration vector fields $Y_i(t) = \sigma_{s^i}(0,t)$, $t \in I$. Then

$$Y_i(0) = \alpha^{(i)}(0) = V_i(0), \qquad i = 1, \ldots, n$$

and

$$
\begin{aligned}
Y_i'(0) &= \sigma_{s^i t}(0,0) = \sigma_{t s^i}(0,0) + \sum_{l=1}^{i-1} (R(\sigma_s, \sigma_t)\sigma_{s^l})_{s^{i-l-1}}(0,0) \\
&= Z^{(i)}(0) + G_{i-1}(V_1(0), \ldots, V_{i-1}(0), \gamma'(0), \ldots, V_{i-1}'(0)) = V_i'(0).
\end{aligned}
$$

Since $(Y_1, \ldots, Y_n)$ is an nth order Jacobi field by Proposition 2.5. we find that $Y^n = V^n$ according to Proposition 2.4. The theorem follows. □

4. Geometric derivatives

Let $F : (N,h) \to (M,g)$ denote a smooth map between semi-Riemannian manifolds. We shall define the jth geometric derivative of F

$$\mathrm{d}^j F : \bigoplus_{i=1}^{j} TN \to TM.$$

To this end let $v_1, \ldots, v_j \in T_pN$. There exists a smooth curve α in N with

$$\alpha^{(i)}(0) = v_i, \qquad i = 1, \ldots, j.$$

Let $\gamma = F \circ \alpha$ and define

$$\mathrm{d}^j F(v_1, \ldots, v_j) = \gamma^{(j)}(0).$$

Verify by induction that in local coordinates

$$\gamma^{(i)} = \frac{\partial^i F}{\partial x_{j_1} \ldots \partial x_{j_2}} \frac{\mathrm{d}\alpha^{j_1}}{\mathrm{d}s} \cdots \frac{\mathrm{d}\alpha^{j_i}}{\mathrm{d}s} + h_i(\alpha, \ldots, \alpha^{(i)}), \qquad i = 1, \ldots, j$$

for some smooth functions

$$h_i : U \times \bigoplus_{k=1}^{i} \mathbb{R}^n \to \mathbb{R}^m$$

showing that $\mathrm{d}^j F$ is well defined and smooth.

Proposition 4.1. *Let $p \in M$ and $x \in \mathbb{D}(\exp_p) \subset T_pM$. For $v_1, \ldots, v_j \in T_xT_pM$, we have*

$$V_j(1) = \mathrm{d}^j \exp_p(v_1, \ldots, v_j),$$

where V^j is the unique nth order Jacobi field along γ_x such that

$$V_i(0) = 0, \quad V_i'(0) = G_{i-1}(0, \ldots, 0, \gamma'(0), \ldots, V_{i-1}'(0)) + v_i, \qquad i = 1, \ldots, j.$$

Proof. Define

$$\sigma(s,t) = \exp\left(t\left(x + \sum_{i=1}^{j} \frac{1}{i!} s^i v_i\right)\right), \qquad (s,t) \in I \times J,$$

which is a geodesic variation of γ_x. Then $V^j(t) = (\sigma_s(t,0), \ldots, \sigma_{s^j}(t,0))$ is an nth order Jacobi field along γ_x with

$$V^j(1) = (\mathrm{d}^1 \exp_p(v_1), \ldots, \mathrm{d}^j \exp_p(v_1, \ldots, v_j)).$$

Also

$$\begin{aligned} V_i'(0) &= \sigma_{ts^i}(0,0) + G_{i-1}(0, \ldots, 0, \gamma'(0), \ldots, V_{i-1}'(0)) \\ &= v_i + G_{i-1}(0, \ldots, 0, \gamma'(0), \ldots, V_{i-1}'(0)). \end{aligned}$$

The proposition follows. □

5. *m* geodesics

We start with:

Definition 5.1. An m geodesic, $m \geq 2$, is a smooth curve $\alpha : I \to M$ such that $\alpha^{(m)} = 0$.

Then:

Proposition 5.2. *Given $v_1, \ldots, v_{m-1} \in T_pM$, then there exists a unique m geodesic $\alpha : I \to M$ such that*

$$\alpha^{(i)}(0) = v_i, \quad i = 1, \ldots, m-1.$$

Proof. Let (U, ϕ) denote a chart around p. Verify by induction that for all $j \geq 2$ there exist smooth functions

$$h_j : \phi(U) \times (\mathbb{R}^n)^{j-1} \to (\mathbb{R}^n)^j$$

such that

$$\beta^{(j)}(0) = \left\{ \frac{\mathrm{d}^j \beta^k}{\mathrm{d}t^j}(0) + h_j^k(\beta(0), \frac{\mathrm{d}\beta}{\mathrm{d}t}(0), \dots, \frac{\mathrm{d}^{j-1}\beta}{\mathrm{d}t^{j-1}}(0)) \right\} \partial_k$$

for all smooth curves $\beta : I \to M$ through $\beta(0) \in U$. Define the vector field

$$X : \phi(U) \times (\mathbb{R}^n)^{m-1} \to (\mathbb{R}^n)^m,$$
$$X(u_1, \dots, u_m) = (u_2, \dots, u_m, -h_m(u_1, \dots, u_m)).$$

Now the proposition amounts to the existence and uniqueness of a local flow for the smooth vector field X.

There is a unique m geodesic $\gamma_{v_1,\dots,v_{m-1}} : J \to M$ such that:
(1) $\gamma'_{v_1,\dots,v_{m-1}}(0) = v_1, \dots, \gamma^{(m-1)}_{v_1,\dots,v_{m-1}}(0) = v_{m-1}$.
(2) If $\alpha : J_* \to M$ is an m geodesic with

$$\alpha'(0) = v_1, \dots, \alpha^{(m-1)}(0) = v_{m-1},$$

then $J_* \subset J$ and α is equal to the restriction of $\gamma_{v_1,\dots,v_{m-1}}$ to J_*.

$\gamma_{v_1,\dots,v_{m-1}}$ is denoted the maximal m geodesic satisfying (1).

Define

$$N^m = \bigoplus_{i=1}^{m-1} TM.$$

We can now define a smooth vector field X on N^m by

$$X(v_1, \dots, v_{m-1}) = [\gamma'_{v_1,\dots,v_{m-1}} + \dots + \gamma^{(m-1)}_{v_1,\dots,v_{m-1}}]_0.$$

X is the m geodesic spray.

Now let $\alpha : I \to M$ denote an m geodesic. The m Jacobi differential equation is the linear differential equation

$$Y^{(m)} = \sum_{i=1}^{m-1} \sum_{\sum_{p=1}^{4} k_p = m-1-i} R_{t^{k_1}}(Y^{(k_2)}, \alpha^{(1+k_3)})\alpha^{(i+k_4)}.$$

A smooth vector field $Y : I \to TM$ along α satisfying this differential equation is called an m Jacobi field along α. □

Proposition 5.3. *Given $w_0, \dots, w_{m-1} \in T_{\alpha(0)}M$, then there exists a unique m Jacobi field Y along α such that*

$$Y^{(i)}(0) = w_i, \quad i = 0, \dots, m-1.$$

Example 5.4. $M = \mathbb{R}^n_\nu$. For $v_1, \ldots, v_{m-1} \in T_{v_0}M$ we have

$$\gamma_{v_1,\ldots,v_{m-1}}(t) = \sum_{i=0}^{m-1} \frac{1}{i!} t^i v_i, \qquad t \in \mathbb{R}.$$

If $A_1, \ldots, A_{m-1}$ are parallel vector fields along α, then

$$Y(t) = \sum_{i=0}^{m-1} t^i A_i(t)$$

is an m Jacobi field.

Proposition 5.5. *The variation vector field of a smooth m geodesic variation*

$$x : I \times J \to M$$

is an m Jacobi field.

Let $\gamma : I \to M$ denote a nonconstant geodesic and $m \geq 3$.

Proposition 5.6. *For all $\epsilon \in I \cap \mathbb{R}_+$ there exists an m Jacobi field $Y \neq 0$ along γ such that $Y(0) = 0$, $Y(\epsilon) = 0$.*

Proof. Define

$$\beta(t) = \gamma \circ h(t), \quad h(t) \in I; \qquad h(t) = \sum_{i=0}^{m-1} \frac{1}{i!} a_i t^i$$

for suitable real constants a_i. Notice that

$$\beta^{(i)}(t) = \gamma' \circ h(t) h^{(i)}(t), \quad h(t) \in I, \quad i = 1, \ldots, m-1,$$

hence $\beta^{(m)} \equiv 0$, so β is an m geodesic with $\beta^{(i)}(0) = a_i \gamma'(0)$.

Define

$$L^m(\epsilon) = \{m \text{ Jacobi field } Y || \gamma' \mid Y(0) = 0, Y(\epsilon) = 0\}.$$

Our aim is to show that this vector space is nontrivial. To this end define smooth m geodesic variations

$$x^i(s, t) = \gamma(s(t^{i-1}\epsilon - t^i) + t), \quad i = 2, \ldots, m-1$$

with variation vector fields

$$Y_i(t) = \frac{\partial x^i}{\partial s}(0, t) = \gamma'(t) t^{i-1}(\epsilon - t).$$

They are m Jacobi fields in view of Proposition 5.5. with

$$Y_i(0) = 0, \qquad Y_i(\epsilon) = 0.$$

The proposition follows. □

Corollary 5.7. $\dim L^m(\epsilon) = m - 2$.

Proof. $Y_2, \ldots, Y_{m-1}$ are clearly linearly independent. Define

$$Y_m = \operatorname{span}\{Y_0 = \gamma', Y_1 = t\gamma'\}$$

and verify that

$$L^m(\epsilon) \oplus Y_m = \{Y\ m \text{ Jacobi field} \| \gamma'\}.$$

The right-hand side is an m-dimensional vector space and Y_m is two dimensional, hence the corollary. □

Corollary 5.8. *When M is flat, then*

$$\dim\{m \text{ Jacobi field } Y \mid Y(0) = 0, Y(\epsilon) = 0\} = n(m-2),$$

where $\epsilon \in I \cap \mathbb{R}_+$.

Now let P denote a smooth semi-Riemannian submanifold of M of dimension $\dim P \geq 1$.

Definition 5.9. An m geodesic $\sigma : I \to M$ is m normal to P provided

$$\sigma^{(i)}(0) \perp T_{\sigma(0)}P, \quad i = 1, \ldots, m-1.$$

An m geodesic σ, m normal to P with $\mathbb{R}_+ \subset I$ gives rise to linear maps

$$S_{\sigma^{(i)}(0)} : T_{\sigma(0)}P \to T_{\sigma(0)}P, \quad i = 1, \ldots, m-1$$

defined by

$$S_{\sigma^{(i)}(0)}(v) = -\Pi(v, \sigma^{(i)}(0)), \quad v \in T_{\sigma(0)}P,$$

where Π is the tensor:

$$\Pi(X, Y) = \tan \nabla_X Y, \quad X \in \Xi(P), \quad Y \in \Xi(P)^\perp,$$

$\Xi(P)$ denoting the space of smooth vector fields in P and $\Xi(P)^\perp$ denoting the space of smooth vector fields along the inclusion map of P in M and orthogonal to P.

For $b \in \mathbb{R}$ define a linear map $L_b : T_{\sigma(0)}P \to T_{\sigma(0)}P$ by

$$L_b(v) = v - \sum_{i=1}^{m-1} \frac{1}{i!} b^i S_{\sigma^{(i)}(0)}(v).$$

It gives rise to the polynomial

$$Q(b) = \det L_b.$$

The multiplicity of $b \in \mathbb{R}$ as a root in Q is denoted $\alpha(b)$, the algebraic multiplicity of b. The index of a linear map L is the number of eigenvalues with negative real part. It is denoted index L.

An m Jacobi field V is tangential provided

$$V^{(i)}(0) + \sum_{k=1}^{i-1} (R(V, \sigma')\sigma^{(k)})_{t^{i-1-k}}(0) \in T_p P$$

for all $i = 1, \ldots, m-1$.

Now let M denote a flat manifold, which is m geodesically complete.

Theorem 5.10. *The space of tangential m Jacobi fields arising as variation vector fields of smooth variations of σ through m geodesics m normal to P and vanishing at $b > 0$ is a vector space $V(b)$ and*

$$\dim V(b) \le \alpha(b).$$

If M is Riemannian then

$$\mathrm{index} - S_{\sigma^{(m-1)}(0)} \le \sum_{b \in \mathbb{R}_+} \dim V(b).$$

Proof. We claim that in a possibly nonflat semi-Riemannian manifold M an m Jacobi field V on σ is the variation vector field of a smooth variation x of σ through m geodesics m normal to P iff

$$\begin{aligned} & V(0) \in T_{\sigma(0)} P, \\ & \tan \left\{ V^{(i)}(0) + \sum_{k=1}^{i-1} (R(\sigma', V)\sigma^{(k)})_{t^{i-1-k}}(0) \right\} \\ & \quad = \Pi(V(0), \sigma^{(i)}(0)), \qquad i = 1, \ldots, m-1. \end{aligned} \tag{5.1}$$

If V is the variation vector field of such an $x : I \times J \to M$ define

$$Z_i(s) = x_{t^i}(0, s), \qquad s \in J, \; i = 1, \ldots, m-1.$$

It is a smooth vector field along $\alpha(s) = x(0, s)$, $s \in J$ orthogonal to P. Also

$$V^{(i)}(0) = Z_i'(0) - \sum_{k=1}^{i-1} (R(\sigma', V)\sigma^{(k)})_{t^{i-1-k}}(0, 0).$$

Since $\alpha' \in TP$ (5.1) follows.

If V is an m Jacobi field along σ satisfying (5.1), let α be a smooth curve in P with $\alpha'(0) = V(0)$. Also let A_i and B_i denote normal parallel vector fields along α with

$$\begin{aligned} & A_i(0) = \sigma^{(i)}(0), \qquad i = 1, \ldots, m-1, \\ & B_i(0) = \mathrm{nor}\{V^{(i)}(0) + \sum_{k=1}^{i-1} (R(\sigma', V)\sigma^{(k)})_{t^{i-1-k}}(0)\}. \end{aligned}$$

Here we agree that a sum with top index strictly smaller than the bottom index is zero. The vector fields above give rise to normal vector fields

$$Z_i(v) = A_i(v) + vB_i(v), \qquad i = 1, \ldots, m-1$$

and an m geodesic variation

$$x(s,t) = \gamma_{Z_1(s),\ldots,Z_{m-1}(s)}(t), \qquad s \in J, \quad t \in I$$

of σ through m geodesics m normal to P. Now

$$\begin{aligned} x_{t^i s}(0,0) &= Z_i'(0) = A_i'(0) + B_i(0) \\ &= \Pi(V(0), \sigma^{(i)}(0)) + \operatorname{nor}\left\{ V^{(i)}(0) + \sum_{k=1}^{i-1} (R(\sigma', V)\sigma^{(k)})_{t^{i-1-k}}(0) \right\} \\ &= V^{(i)}(0) + \sum_{k=1}^{i-1} (R(\sigma', V)\sigma^{(k)})_{t^{i-1-k}}(0). \end{aligned}$$

If $Y(t) = x_s(0,t)$, $t \in I$ then

$$Y(0) = x_s(0,0) = \alpha'(0) = V(0).$$

and

$$Y^{(i)}(0) = x_{t^i s}(0,0) - \sum_{k=1}^{i-1} (R(x_t, x_s)x_{t^k})_{t^{i-1-k}}(0,0) = V^{(i)}(0).$$

By Propositions 5.5 and 5.3 the variation vector field of x is V and the claim follows. It follows from the claim that $V(b)$ is a vector space for all $b > 0$. An m, P Jacobi field along σ is an m Jacobi field V satisfying (5.1). $\sigma(b)$, $b \neq 0$, is an m focal point for P along σ provided there exists an m, P Jacobi field $V \neq 0$ along σ with $V(b) = 0$. Now suppose that M is flat.

Our next claim is that $\sigma(b)$, $b > 0$, is an m focal point for P along σ iff $Q(b) = 0$. Also

$$\dim V(b) = \dim \ker L_b.$$

If $Q(b) = 0$ then L_b is singular and there exists a nonzero $v \in \ker L_b$. Take parallel vector fields A_i, $i = 0, \ldots, m-1$, along σ with

$$A_0(0) = v, \qquad A_i(0) = -\frac{1}{i!} S_{\sigma^{(i)}(0)}(v), \qquad i = 1, \ldots, m-1.$$

By Example 5.4

$$V(t) = \sum_{i=0}^{m-1} t^i A_i(t), \qquad t \in I$$

is an m Jacobi field satisfying (5.1). Notice that the parallel vector field

$$Y(t) = \sum_{i=0}^{m-1} b^i A_i(t), \qquad t \in I$$

vanishes identically because $L_b(v) = 0$, hence $V(b) = 0$. It follows that

$$\dim V(b) \geq \dim \ker L_b.$$

On the other hand

$$V(b) \to \ker L_b, \qquad V \mapsto V(0) = v$$

is a linear injection so $\dim V(b) = \dim \ker L_b$. If on the other hand, $V \neq 0$ is an m, P Jacobi field along σ with $V(b) = 0$ then

$$V(t) = \sum_{i=1}^{m-1} t^i A_i(t)$$

for suitable parallel vector fields A_i along σ with

$$V^{(i)}(0) = -\frac{1}{i!} S_{\sigma^{(i)}(0)}(v), \qquad v = V(0).$$

So $v \neq 0$ and $v \in \ker L_b$. Hence $Q(b) = 0$. This verifies the second claim.

Now let $a = \dim V(b)$. If $a = 0$ then $\dim V(b) \leq \alpha(b)$ is obvious, otherwise let $v_1, \dots, v_a, \dots, v_n$ be a basis for $T_{\sigma(0)}M$ such that $v_1, \dots, v_a$ is a basis for $\ker L_b$. The matrix representation for L_s in this basis is denoted $l_{ij}(s)$ so

$$L_s(v_i) = \sum_{j=1}^{n} l_{ji}(s) v_j.$$

Hence $l_{ij}(b) = 0$ for all $j = 1, \dots, a$ and all $i = 1, \dots, n$. From

$$Q(s) = \sum_{\sigma \in S_n} (-1)^{\operatorname{sign} \sigma} l_{1\sigma(1)}(s) \cdots l_{n\sigma(n)}(s).$$

We deduce that

$$Q(b) = 0, \dots, Q^{(a-1)}(b) = 0.$$

We conclude that $a \leq \alpha(b)$ and the first inequality follows.

To prove the second inequality define for a topological space X

$$SP^n(X) = X \times \cdots \times X / \sim,$$

where $x \sim y$ iff there exists $\sigma \in S_n$ such that $x_i = y_{\sigma(i)}$ for all $i = 1, \dots, n$. Denote an equivalence class by $\prod_{i=1}^{n} x_i$. Also define a map

$$T : \mathbb{C}^{n+1} \backslash \{c_n = 0\} = B_{n+1} \to SP^n(\mathbb{C}), \qquad (c_0, \dots, c_n) \mapsto \prod_{i=1}^{n} \lambda_i,$$

where

$$\sum_{i=0}^{n} c_i x^i = c_n \prod_{i=1}^{n} (x - \lambda_i).$$

We claim that this map T is continuous. To see this let

$$\pi_{n+1} : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{C}P^n$$

denote the natural map and define

$$A = \pi_2(B), \qquad B = \{(a, b) \in \mathbb{C}^2 \backslash \{0\} \mid a \neq 0\}.$$

The map

$$G : SP^n(A) \to SP^n(\mathbb{C}), \qquad \prod_{i=1}^{n} [a_i, b_i] \mapsto \prod_{i=1}^{n} b_i/a_i$$

is well defined and continuous. The map

$$S : \mathbb{C}P^n \to SP^n(\mathbb{C}P^1), \qquad [c_0, \dots, c_n] \mapsto \prod_{i=1}^{n} [a_i, b_i],$$

where

$$\prod_{i=1}^{n} (a_i x - b_i) = \sum_{i=0}^{n} c_i x^i$$

is continuous by [1]. Let $A_{n+1} = \pi_{n+1}(B_{n+1})$. Notice that

$$T = G \circ S \mid_{A_{n+1}} \circ \pi_{n+1}$$

is continuous as claimed.

Since T is continuous index $L_b = 0$ near $b = 0$. Let $\lambda_1(b), \dots, \lambda_n(b)$ denote the n real eigenvalues of the self-adjoint linear map L_b. Also let $0 < b_1 < \cdots < b_j$ denote the positive zeroes of Q:

$$\dim\ker L_{b_i} = \alpha_i \geq 1,$$
$$\dim\ker L_b = 0, \qquad b \neq b_1, \dots, b_j, \quad b \geq 0.$$

If there exists an $i \in \{1, \dots, n\}$ and $b \in]0, b_1]$ such that $\lambda_i(b) < 0$ then by continuity of T above there exists an i and a $b_* \in]0, b[$ such that $\lambda_i(b_*) = 0$, a contradiction. Therefore

$$\text{index}\, L_b = 0, \qquad b \in [0, b_1].$$

Similarly if index $L_b > \alpha_1$ for a $b \in]b_1, b_2]$ there exists $\beta_{i_1}, \dots, \beta_{i_k}$, $k > \alpha_1$ such that

$$\lambda_{\beta_{i_1}}(b) < 0, \dots, \lambda_{\beta_{i_k}}(b) < 0.$$

By continuity of T this implies that there exists $\beta_{i_1}, \dots, \beta_{i_k}$ such that

$$\lambda_{\beta_{i_1}}(b_1) = \cdots = \lambda_{\beta_{i_k}}(b_1) = 0.$$

A contradiction and

$$\text{index}\, L_b \leq \alpha_1, \qquad b \in]b_1, b_2].$$

By induction

$$\text{index}\, L_b \leq \alpha_1 + \cdots + \alpha_k, \qquad b \in]b_k, b_{k+1}]$$

for $k = 1, \dots, j-1$ and

$$\text{index}\, L_b \le \alpha_1 + \cdots + \alpha_j, \qquad b > b_j.$$

Now let A denote a subspace of dimension index $-\, S_{\sigma^{(m-1)}(0)}$ on which the scalar product

$$(v, w) \mapsto \langle -S_{\sigma^{(m-1)}(0)} v, w\rangle = b(v, w)$$

is negative definite. On the unit sphere $S(A)$ in A we have

$$-b(v, v) > c, \qquad v \in S(A)$$

for some constant $c > 0$. Hence

$$\langle L_b(v), v\rangle = b^{m-1}\left(-\langle S_{\sigma^{(m-1)}(0)} v, v\rangle \frac{1}{(m-1)!} - \sum_{j=0}^{m-2} \frac{1}{j!} b^{j-(m-1)} \langle S_{\sigma^{(j)}(0)} v, v\rangle \right) < 0$$

for all b greater than some $b_0 > b_j$. Hence for $b > b_0$ we have

$$\text{index} - S_{\sigma^{(m-1)}(0)} \le \text{index}\, L_b \le \alpha_1 + \cdots + \alpha_j = \sum_{b \in \mathbb{R}_+} \dim V(b).$$

The theorem follows. $\square$

Corollary 5.11. *The space of m Jacobi fields arising as variation vector fields of smooth variations of σ through m geodesics m normal to P and vanishing at $b > 0$ is a vector space $W(b)$ and*

$$\dim W(b) = \dim V(b) + (m-2)(n-p), \qquad \dim P = p,$$
$$\text{index} - S_{\sigma^{(m-1)}(0)} \le \sum_{b \in \mathbb{R}_+} (\dim W(b) - (m-2)(n-p)).$$

Proof. $W(b)$ is a vector space due to the first claim in Theorem 5.10. First notice that

$$W(b) = W^{\mathrm{T}}(b) \oplus W^{\perp}(b),$$

where

$$\begin{aligned} W^{\mathrm{T}}(b) = \{&Y\ m, P \text{ Jacobi field along } \sigma \mid Y^{(i)}(0) \in T_p P, \\ &i = 0, \dots, m-1,\ Y(b) = 0\}, \\ W^{\perp}(b) = \{&Y\ m, P \text{ Jacobi field along } \sigma \mid Y^{(i)}(0) \in T_p P^{\perp}, \\ &i = 0, \dots, m-1,\ Y(b) = 0\}. \end{aligned}$$

To see this let $Y \in W(b)$ that is

$$Y(t) = \sum_{i=0}^{m-1} t^i A_i(t)$$

for suitable parallel vector fields $A_0, \dots, A_{m-1}$. Write uniquely

$$A_i(0) = v_i + w_i, \quad v_i \in T_pP, \quad w_i \in T_pP^\perp$$

and let A_i^1 and A_i^2 denote the parallel vector fields with

$$A_i^1(0) = v_i, \quad A_i^2(0) = w_i, \qquad i = 0, \dots, m-1.$$

Then

$$Y(t) = \sum_{i=0}^{m-1} t^i A_i^1(t) + \sum_{i=0}^{m-1} t^i A_i^2(t),$$

where the first sum is an m, P Jacobi field in $W^{\mathrm{T}}(b)$ and the second sum is an m, P Jacobi field in $W^\perp(b)$. Let $E_{p+1}, \dots, E_n$ denote parallel vector fields along σ such that

$$\operatorname{span}\{E_{p+1}(0), \dots, E_n(0)\} = T_pP^\perp.$$

Then

$$V_i^j(t) = t^{i-1}(b-t)E_j(t), \quad j = p+1, \dots, n, \quad i-1 = 1, \dots, m-2$$

is a basis for $W^\perp(b)$ and the corollary follows, since $V(b) = W^{\mathrm{T}}(b)$. □

Now let $\alpha : I \to M$ denote an m geodesic in an arbitrary semi-Riemannian manifold. Given $\epsilon > 0$ in I define

$$Y^m(\epsilon) = \{Y m \text{ Jacobi field along } \alpha \mid Y(0) = 0,\ Y(\epsilon) = 0\}.$$

Proposition 5.12. *There exists* $\epsilon > 0$ *such that*

$$\dim Y^m(t) = n(m-2)$$

for all $t \in]0, \epsilon[$.

Proof. Notice that

$$\exp_p^m(x_1, 0, \dots, 0) = \exp_p(x_1), \quad x_1 \in \mathbb{D}(\exp_p),$$

where

$$\exp_p^m(v_1, \dots, v_{m-1}) = \gamma_{v_1,\dots,v_{m-1}}(1),$$

whenever 1 belongs to the domain of definition of $\gamma_{v_1,\dots,v_{m-1}}$. Letting $x_i = \alpha^{(i)}(0)$, we deduce that there exists $\epsilon > 0$ such that

$$\mathrm{d}_1 \exp_p^m(tx_1, \dots, t^{m-1}x_{m-1})$$

is an isomorphism for all $t \in [0, \epsilon[$, since $\mathrm{d}\exp_p(0)$ is an isomorphism.

We claim that there exist linear mappings

$$h_i : \bigoplus_{k=1}^{i-1} T_pM \to T_pM$$

such that

$$V(t) = \mathrm{d}\exp_p^m(tx_1, \ldots, t^{m-1}x_{m-1})(tv_1, \ldots, t^{m-1}v_{m-1}),$$

where V is the m Jacobi field along α such that $V(0) = 0$, $V'(0) = v_1$ and

$$V^{(i)}(0) = v_i + h_i(v_1, \ldots, v_{i-1}), \qquad i = 2, \ldots, m-1.$$

To see this define

$$\tilde{x} : [0, 1] \times I \to N^m, \qquad \tilde{x}(t, s) = (t(x_1 + sv_1), \ldots, t^{m-1}(x_{m-1} + sv_{m-1}))$$

and

$$x(t, s) = \exp_p^m(\tilde{x}(t, s)).$$

The variation vector field of this variation is

$$V(t) = \mathrm{d}\exp_p^m((tv_1, \ldots, t^{m-1}v_{m-1})_{(tx_1, \ldots, t^{m-1}x_{m-1})}).$$

Clearly, $V(0) = 0$, $V'(0) = v_1$ and

$$V''(0) = R(x_s, x_t)x_t(0, 0) + x_{tts}(0, 0) = (x_2 + sv_2)_s(0) = v_2$$

so $h_2(v_1) = 0$. Assuming the existence of h_j, $j = 1, \ldots, i-1 \le m-2$ compute

$$\begin{aligned}
x_{st^i}(0, 0) &= x_{t^is}(0, 0) + \sum_{k=1}^{i-1} (R(x_t, x_s)x_{t^k})_{t^{i-1-k}}(0, 0) \\
&= (x_i + sv_i)_s(0) - \sum_{k=1}^{i-1} \sum_{\sum k_p = i-1-k} R_{t^{k_1}}(x_{st^{k_2}}, x_{t^{k_3+1}})x_{t^{k+k_4}}(0, 0) \\
&= v_i - \sum_{k=1}^{i-1} \sum_{\sum k_p = i-1-k} R_{t^{k_1}}(v_{k_2} + h_{k_2}(v_1, \ldots, v_{k_2-1}), x_{k_3+1})x_{k+k_4} \\
&= v_i + h_i(v_1, \ldots, v_{i-1}).
\end{aligned}$$

The claim follows. Now define an $n(m-1)$-dimensional vector space

$$Y^{m-1} = \{Y\, m \text{ Jacobi field along } \gamma \mid Y(0) = 0\}$$

and linear isomorphisms

$$G : Y^{m-1} \to T_pM^{m-1}, \qquad Y \mapsto (Y'(0), \ldots, Y^{(m-1)}(0))$$

and for $t > 0$

$$H_t : T_pM^{m-1} \to T_pM^{m-1},$$
$$(tv_1, \dots, t^{m-1}v_{m-1}) \mapsto (v_1, v_2 + h_2(v_1), \dots, v_{m-1} + h_{m-1}(v_1, \dots, v_{m-2})).$$

Then from what we have seen

$$V(t) = \mathrm{d}\exp_p^m(tx_1, \dots, t^{m-1}x_{m-1})(H_t^{-1} \circ G(V)) = K_t(V).$$

The rank of the linear map K_t is n for $t \in]0, \epsilon[$ hence

$$\dim Y^m(t) = \dim\ker K_t = n(m-1) - n = n(m-2).$$

The proposition follows. □

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